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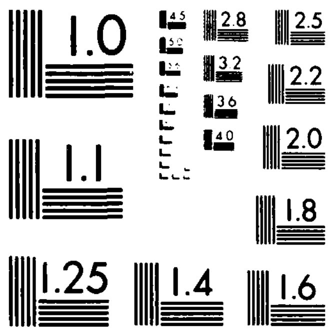


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# STATISTICAL ANALYSIS OF AUTOREGRESSIVE SPECTRAL ESTIMATES FOR NOISE CORRUPTED AUTOREGRESSIVE SERIES

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Prepared for
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Code 411

**Naval Ocean Systems Center**
San Diego, California 92152





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Estimation of the spectral density function for a gaussian distributed autoregressive series from observations of a noise corrupted version is considered when the order of the autoregressive series is assumed to be known. When the high-order Yule-Walker equation estimates of the autoregressive parameters are used to form the spectral density estimate, it is shown that the estimate is weakly consistent and asymptotically normal with zero mean and finite variance. A closed form expression for the asymptotic variance is developed and the expression is analyzed for the first-order AR series case.


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## CONTENTS



## ILLUSTRATIONS

# I. INTRODUCTION

This report considers the problem of estimating the spectral density of a discrete-time autoregressive (AR) series from observations of a noise corrupted version. The spectral density estimate is based on the high-order Yule-Walker equation estimates of the AR parameters. Under the assumption that the order of the autoregressive series is known, the limiting distribution of the spectral density estimate is normal with mean zero and finite variance. The mean and variance of the limiting distribution, for the noise corrupted case, have not previously been evaluated.

The problem of AR parameter estimation for the noise corrupted case was previously considered by Walker (reference 1), Pagano (reference 2), and Lee (reference 3). Walker was the first to consider this problem; he evaluated the asymptotic efficiency and variance for the parameter estimates of a first order series. Pagano proved that an equivalent model for an autoregressive series plus noise is an autoregressive-moving-average (ARMA) model. Through the use of nonlinear regression methods, he developed strongly consistent, efficient parameter estimates. Lee recently examined the multivariate noise corrupted case and proved that the multivariate parameter estimates are strongly consistent and asymptotically normal.

The organization of this paper is as follows: In Section II, the form of the spectral density and the AR parameter estimator for the noise corrupted case is established. In Section III proof is offered that the limiting distribution of the AR spectral density estimate is normal with mean zero and the asymptotic variance expression is evaluated. In Section IV, the variance expression for the first-order Markov series (as an example) is evaluated.

# II. PRELIMINARIES

Let $\{Y_n\}_{n=-\infty}^{\infty}$ be a discrete parameter time series satisfying the following assumption:

Assumption A: The series $\{Y_n\}$ consists of the sum of an autoregressive series $\{X_n\}$ of known order p and a noise series $\{W_n\}$. The AR series $\{X_n\}$ is generated (or modeled) by

$$X_n - a_1 X_{n-1} - \cdots - a_p X_{n-p} = \varepsilon_n \tag{1}$$

and

(i) $\{\varepsilon_n\}$ is stationary independent identically distributed $N(0,\sigma_\varepsilon^2)$

(ii) $\{W_n\}$ is stationary independent identically distributed $N(0,\sigma_W^2)$

(iii) $\{\varepsilon_n\}$ and $\{W_n\}$ are uncorrelated

The parameter set $\{a_j\}_{j=1}^{p}$ is referred to as the AR parameter set.

Assumption B: The AR parameters are constrained such that the zeros of the polynomial

$$A^p(z) = 1 - \sum_{j=1}^{p} a_j z^j \tag{2}$$

lie outside of the unit circle on the complex z-plane.

Under Assumption B the AR series is stationary. It was assumed that the noise is wide-sense stationary; thus, the spectral density function for the noise corrupted series Y can be written as

$$\phi_Y(\lambda) = \frac{\sigma_W^2}{2\pi} + \frac{\sigma_\varepsilon^2}{2\pi A^p(e^{i\lambda})A^p(e^{-i\lambda})} \quad . \tag{3}$$

Walker (reference 1) and Pagano (reference 2) showed that the AR plus noise series can be expressed as an ARMA series. We express the noise corrupted series Y by

$$Y_n - a_1 Y_{n-1} - \cdots - a_p Y_{n-p} = \varepsilon_n + w_n - a_1 w_{n-1} - \cdots - a_p w_{n-p} \quad . \tag{4}$$

Let the covariance sequence of the series Y be $\{r_k\}$, where $r_k = E[Y_n Y_{n-k}]$. Multiplying (4) through by $Y_{n-k}$ and taking expectations term by term we obtain the Yule-Walker (Y-W) equations:

$$r_o - a_1 r_1 - \cdots - a_p r_p = \sigma_\varepsilon^2 + \sigma_w^2 \qquad (k = 0) \tag{5}$$

$$r_k - a_1 r_{k-1} - \cdots - a_p r_{k-p} = -a_k \sigma_w^2 \qquad (1 \leq k \leq p) \tag{6}$$

$$r_k - a_1 r_{k-1} - \cdots - a_p r_{k-p} = 0 \qquad (p+1 \leq k \leq 2p) \ . \tag{7}$$

The set of p equations of (7) are often referred to as the high-order Yule-Walker equations. We express this set of equations in matrix form as

$$\underline{\Gamma}_p \, \underline{a} = \underline{R}_{p+1} \tag{8}$$

where the $(p \times p)$ covariance matrix $\underline{\Gamma}_p$ is defined by

$$\underline{\Gamma}_p \triangleq \begin{bmatrix} r_p & r_{p-1} & \cdots & r_1 \\ r_{p+1} & r_p & \cdots & r_2 \\ \vdots & \vdots & & \vdots \\ r_{2p-1} & r_{2p-2} & \cdots & r_p \end{bmatrix} \ . \tag{9}$$

and the $(p \times 1)$ vectors $\underline{a}$ and $\underline{R}_{p+1}$ are defined by

$$\underline{a}^T = [a_1, a_2, \cdots, a_p]$$

$$\underline{R}_{p+1}^T = [r_{p+1}, r_{p+2}, \cdots, r_{2p}] \ .$$

Given a finite set of observations of the noise corrupted series Y, that is $\{Y_n\}_{n=1}^N$ $N > 2p$, we estimate the covariance sequence $\{r_k\}$ using

$$\hat{r}_k = \begin{cases} \dfrac{1}{N} \displaystyle\sum_{n=1}^{N-|k|} Y_n Y_{n+|k|} & |k| \leq N-1 \\ 0 & |k| > N-1 \end{cases} \tag{10}$$

When the covariances $r_k$ of the matrix $\underline{\Gamma}_p$ and the vector $\underline{R}_{p+1}$ are replaced by their corresponding estimates of (10), the estimated matrix and vector will be denoted by $\hat{\underline{\Gamma}}_p$ and $\hat{\underline{R}}_{p+1}$, respectively. The high-order Y-W equations (8) can be expressed in terms of the estimated covariances as

$$\underline{\hat{\Gamma}}_p \, \underline{\hat{a}} = \underline{\hat{R}}_{p+1} \quad . \tag{11}$$

The solution of (11) in terms of $\underline{\hat{a}}$ provides the high-order Y-W equation estimate of the AR parameters.

In order to estimate the AR spectral density we require estimates of the AR parameters such as those formed by (11) and an estimate of $\sigma_\varepsilon^2$. In the noise free case, given estimates of the covariances $\{r_k\}$ and AR parameters $\{a_j\}_{j=1}^p$ (5) can be used to estimate $\sigma_\varepsilon^2$. For the noise corrupted case (5) will provide an estimate of $\sigma_\varepsilon^2 + \sigma_w^2$, thus, one of the equations of (6) must also be used to estimate $\sigma_w^2$. Using this approach, with covariance estimates of (10) and estimates of the AR parameters of (11) we have

$$\hat{\sigma}_\varepsilon^2 = - \sum_{j=0}^{p} \hat{a}_j \, \hat{r}_j - (1/\hat{a}_p) \sum_{j=0}^{p} \hat{a}_j \, \hat{r}_{p-j} \tag{12}$$

where $a_o = -1$ and $a_p \neq 0$.

In the subsequent development of asymptotic statistical properties for the parameter and spectral density estimates, we make use of the following vectors and matrices:

$$\underline{R}^T \triangleq [r_1, r_2, \cdots, r_{2p}]$$

$$\underline{R}^T_p \triangleq [r_p, r_{p+1}, \cdots, r_{2p-1}]$$

$$\underline{U}_{nm} \triangleq \{u_{k,j}\} \quad k = n, n+1, \cdots, 2p$$

$$j = m, m+1, \cdots, 2p$$

$$u_{kj} \triangleq e^{i(k+j)\lambda} + e^{i(k-j)\lambda}$$

$$\underline{0} = [0, 0, \cdots, 0] \ .$$

## III. ASYMPTOTIC PROPERTIES

### A. AR Parameter Estimate Statistics

Define the AR parameter vector $\underline{\theta}^T$ by

$$\underline{\theta}^T \triangleq [\sigma_\varepsilon^2, a_1, \cdots, a_p] \ . \tag{13}$$

Our present goal is to establish the asymptotic distribution for estimates of the AR parameter vector. First, we present the asymptotic distribution of the covariance estimates of (10) as established by Brillinger (reference 4).

Theorem 1: For the AR plus noise series Y, under Assumptions A and B, the elements of the covariance vector

$$N^{\frac{1}{2}} (\hat{\underline{R}} - \underline{R})$$

are asymptotically jointly multivariate normal with mean zero and covariance

$$\lim_{N\to\infty} E \{N^{\frac{1}{2}} (\hat{\underline{R}} - \underline{R}), N^{\frac{1}{2}} (\hat{\underline{R}} - \underline{R})^T\} = 2\pi \int_{-\pi}^{\pi} \underline{U}_{11}\phi^2(\lambda)d\lambda \ . \tag{14}$$

The following lemma establishes the existence of a random vector $\underline{Z}$ that is equivalent in distribution to the high order Y-W AR parameter estimate vector $(\hat{\underline{a}} - \underline{a})$. In preparation for this lemma we define the matrix $\underline{D}$ by

$$\underline{D} \triangleq \begin{bmatrix} -a_p & -a_{p-1} & \cdots & -a_1 & 1 & 0 & \cdots & & 0 \\ 0 & -a_p & & \cdots & -a_1 & 1 & 0 & \cdots & 0 \\ \vdots & & & & & & & & \vdots \\ 0 & \cdots & 0 & -a_p & & \cdots & & -a_1 & 1 \end{bmatrix}$$

Lemma 1: For the AR plus noise series Y there exists a $p \times 2p$ matrix $\underline{D}$ and a random vector $\underline{Z}$ such that

$$N^{\frac{1}{2}} (\hat{\underline{a}} - \underline{a}) \sim N^{\frac{1}{2}} \underline{Z} = N^{\frac{1}{2}} [\underline{\Gamma}_p^{-1}\underline{D}(\hat{\underline{R}} - \underline{R})] \tag{15}$$

where ~ indicates that the limit distribution as $N \to \infty$ is identical for both random vectors.

Proof: Define the vector $\underline{V}$ by

$$\underline{V} \triangleq \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_p \end{bmatrix} \triangleq N^{\frac{1}{2}} (\hat{\underline{\Gamma}}_p^{-1} - \underline{\Gamma}_p^{-1})(\underline{\hat{R}}_{p+1} - \hat{\underline{\Gamma}}_p \underline{a}) \ .$$

Since $\underline{\Gamma}_p$ is positive definite and $\hat{\underline{\Gamma}}_p \xrightarrow[N\to\infty]{P} \underline{\Gamma}_p$, element by element, it follows that $\hat{\underline{\Gamma}}_p^{-1} \xrightarrow[N\to\infty]{P} \underline{\Gamma}_p^{-1}$, and

$$v_j \xrightarrow[N\to\infty]{P} 0 \qquad j = 1, 2, \cdots, p \ . \tag{16}$$

Let $(\Omega, \mathcal{F}, P)$ be the underlying probability space. For arbitrary $\varepsilon > 0$ and $N > p$ let

$$\Lambda_{\varepsilon,N} = \{\omega \in \Omega: \ |v_j| < \varepsilon, \ j = 1, 2, \cdots, p\}$$

then for all $\omega \in \Lambda_{\varepsilon,N}$, since $|v_j| < \varepsilon$, we can write

$$\hat{\underline{\Gamma}}_p^{-1} (\underline{\hat{R}}_{p+1} - \hat{\underline{\Gamma}}_p \underline{a}) = (\underline{\hat{a}} - \underline{a}) \ .$$

It follows that

$$\underline{V} = N^{\frac{1}{2}} [(\underline{\hat{a}} - \underline{a}) - \underline{\Gamma}_p^{-1} (\underline{\hat{R}}_{p+1} - \hat{\underline{\Gamma}}_p \underline{a})] \ .$$

for all $\omega \in \Lambda_{\varepsilon,N}$. By (16) we have that for every $\alpha \varepsilon [0,1]$ there exists a $N^*_{\varepsilon,\alpha}$ such that

$$P(\Lambda_{\varepsilon,N}) > 1 - \alpha \qquad N > N^*_{\varepsilon,\alpha} \ .$$

Since the selection of $\varepsilon$ and $\alpha$ is arbitrary we can conclude that

$$N^{\frac{1}{2}}(\underline{\hat{a}} - \underline{a}) - N^{\frac{1}{2}}\Gamma_p^{-1}(\underline{\hat{R}}_{p+1} - \underline{\hat{\Gamma}}_p\underline{a}) \xrightarrow[N\to\infty]{P} \underline{0}$$

By the definition of the matrix $\underline{D}$ and by the high-order Y-W equations (8) and (11) we can write

$$\underline{D}(\underline{\hat{R}} - \underline{R}) = \underline{\hat{R}}_{p+1} - \underline{\hat{\Gamma}}_p\underline{a}$$

and the desired result follows directly. □

We previously established an estimator for the variance $\sigma_\varepsilon^2$, see (12). In the following lemma we establish the existence of an equivalent (in distribution) random variable from which the asymptotic distribution of $\hat{\sigma}_\varepsilon^2$ can be evaluated. In preparation for the lemma we define the vectors:

$$\underline{R}_0^T \triangleq [r_0, r_1, \cdots, r_{2p-1}]$$

$$\underline{\tilde{a}}^T \triangleq [a_p, a_{p-1}, \cdots, a_1] .$$

Lemma 2: For the AR plus noise series Y there exists a random variable $\xi$ such that

$$N^{\frac{1}{2}}(\hat{\sigma}_\varepsilon^2 - \sigma_\varepsilon^2) \sim N^{\frac{1}{2}} \xi = N^{\frac{1}{2}}\underline{H}[\underline{\hat{R}}_0 - \underline{R}_0] \quad . \tag{17}$$

where

$$\underline{H} = -\{[-1, \underline{a}^T, \underline{0}] + (1/a_p) [\underline{\tilde{a}}^T, -1, \underline{0}]\} \quad .$$

*Proof*: By (5), (6), and (12) we write

$$N^{\frac{1}{2}}(\hat{\sigma}_\varepsilon^2 - \sigma_\varepsilon^2) = N^{\frac{1}{2}} \left\{ - \sum_{j=0}^{p} \hat{a}_j \hat{r}_j + \sum_{j=0}^{p} a_j r_j - (1/\hat{a}_p) \sum_{j=0}^{p} \hat{a}_j \hat{r}_{p-j} \right.$$

$$\left. + (1/a_p) \sum_{j=0}^{p} a_j r_{p-j} \right\} .$$

By Gersch (reference 5) we have that the high-order Y-W equation AR parameter estimates converge in probability to the true parameters as $N\to\infty$. Thus, we can write

$$N^{\frac{1}{2}}(\hat{\sigma}_\varepsilon^2 - \sigma_\varepsilon^2) \sim N^{\frac{1}{2}} \left\{ - \sum_{j=0}^{p} \hat{r}_j(\hat{a}_j - a_j) - \sum_{j=0}^{p} a_j(\hat{r}_j - r_j) \right.$$

$$\left. - (1/a_p) \sum_{j=0}^{p} \hat{r}_{p-j} (\hat{a}_j - a_j) - (1/a_p) \sum_{j=0}^{p} a_j(\hat{r}_{p-j} - r_{p-j}) \right\} . \qquad (18)$$

Also, by the convergence in probability result of Gersch (reference 5) we have that $N^{\frac{1}{2}}(\hat{a}_j - a_j) \xrightarrow{P} 0$ as $N\to\infty$ ($j = 1, 2, \cdots, p$); thus, the first and third terms on the right-hand side of (18) converge to zero and the desired result follows directly. □

*Theorem 2*: Under Assumptions A and B the AR parameter estimates converge in distribution to a zero mean normal random vector, that is

$$N^{\frac{1}{2}} (\hat{\underline{\theta}} - \underline{\theta}) \xrightarrow[N\to\infty]{\mathscr{L}} N_{p+1} (\underline{0}, \underline{\Sigma})$$

where

$$\underline{\Sigma} \triangleq \lim_{N\to\infty} N\, E\left[\begin{bmatrix} \xi \\ \underline{Z} \end{bmatrix} \begin{bmatrix} \xi & \underline{Z}^T \end{bmatrix}\right]. \qquad (19)$$

Proof: This result follows directly from the results of Lemmas 1 and 2 and Theorem 1. □

We now proceed to evaluate the terms of $\underline{\Sigma}$. Let

$$\underline{\Sigma} = \left[\begin{array}{c|c} V^2 & \underline{C}^T \\ \hline \underline{C} & \underline{\Phi} \end{array}\right]$$

where $V^2$, $\underline{C}$, and $\underline{\Sigma}$ are defined by (19). For $V^2$ we have

$$V^2 = \lim_{N\to\infty} N\ E[\xi^2] = \lim_{N\to\infty} N\ \underline{H}\ E\ \{[\hat{\underline{R}}_0 - \underline{R}_0][\hat{\underline{R}}_0 - \underline{R}_0]^T\}\underline{H}^T$$

and by Theorem 1 we have

$$V^2 = 2\pi \int_{-\pi}^{\pi} \underline{H}\ \underline{U}_{00}\ \underline{H}^T\ \phi_Y^2(\lambda)d\lambda \quad .$$

After further manipulation (see appendix) we get

$$V^2 = \sigma_w^4\left[\sum_{j=0}^{p} a_j^2 + (2/a_p)\sum_{j=0}^{p} a_j a_{p-j} - 1 + (1/a_p)^2 \sum_{j=0}^{p} a_j^2\right]$$

$$+ \sigma_w^2\sigma_\varepsilon^2\left[\left(2\Big/\sum_{j=0}^{p} a_j^2\right) + 3 + (1/a_p)^2\right]$$

$$+ \sigma_\varepsilon^2\left[\sigma_\varepsilon^2 + r_o + (2/a_p)r_p + (1/a_p)^2 r_o\right] \tag{20}$$

We also have

$$\underline{C}^T = \lim_{N\to\infty} N\ E[\xi\underline{Z}^T] = \lim_{N\to\infty} N\ \underline{H}\ E\{[\hat{\underline{R}}_0 - \underline{R}_0][\hat{\underline{R}} - \underline{R}]\}\ \underline{D}^T \left(\underline{\Gamma}_p^{-1}\right)^T$$

and by Theorem 1 we have

$$\underline{C}^T = 2\pi \int_{-\pi}^{\pi} \underline{H}\ \underline{U}_{01}\ \underline{D}^T \left(\underline{\Gamma}_p^{-1}\right)^T \phi_Y^2(\lambda)d\lambda\ .$$

After further manipulation (see appendix) we get

$$\underline{C}^T = \underline{P}^T\left(\underline{\Gamma}_p^{-1}\right)^T \tag{21}$$

where the vector $\underline{P}$ is defined by

$$\{\underline{P}^T\}_l \triangleq -\sigma_\varepsilon^2 r_{p+l} - \sigma_w^4 \sum_{j=l}^{p} a_j a_{p+l-j} - (1/a_p)\ \sigma_w^4 \sum_{j=0}^{p-l} a_j a_{j+l} - (1/a_p)\ \sigma_\varepsilon^2 r_l$$

$$l = 1, 2, \cdots, p\ .$$

By (15) we get

$$\underline{\Phi} = \lim_{N\to\infty} N\ E[\underline{Z}\ \underline{Z}^T] = \lim_{N\to\infty} N\ \underline{\Gamma}_p^{-1}\underline{D}\ E\{[\hat{\underline{R}} - \underline{R}][\hat{\underline{R}} - \underline{R}]^T\}\underline{D}^T \left(\underline{\Gamma}_p^{-1}\right)^T$$

and by Theorem 1 we can write

$$\underline{\Phi} = 2\pi \int_{-\pi}^{\pi} \underline{\Gamma}_p^{-1}\ \underline{D}\ \underline{U}_{11}\ \underline{D}^T \left(\underline{\Gamma}_p^{-1}\right)^T \phi_Y^2(\lambda)d\lambda\ .$$

After further manipulation (see appendix) we get

$$\underline{\Phi} = \sigma_\varepsilon^2\ \underline{\Gamma}_p^{-1}\ \underline{\Gamma}_o\left(\underline{\Gamma}_p^{-1}\right)^T + \sigma_w^2\ \underline{\Gamma}_p^{-1}\ [\sigma_\varepsilon^2\ \underline{I} + \sigma_w^2\ \underline{Q}]\left(\underline{\Gamma}_p^{-1}\right)^T \tag{22}$$

where $\underline{I}$ is the $p \times p$ identity matrix and $\underline{Q}$ is given by

$$
\underline{Q} \triangleq \begin{bmatrix} \sum_{m=0}^{p} a_m^2 & \sum_{m=0}^{p-1} a_m a_{m+1} & \cdots & \sum_{m=0}^{1} a_m a_{m+(p-1)} \\ \sum_{m=0}^{p-1} a_m a_{m+1} & & & \\ \vdots & & & \vdots \\ \sum_{m=0}^{1} a_m a_{m+(p-1)} & & & \sum_{m=0}^{p} a_m^2 \end{bmatrix} .
$$

B. AR Spectral Density Estimates Statistics

We now proceed to evaluate the limiting distribution of the spectral density estimate for the AR series X formed from observations of the noise corrupted series Y. From (3) we see that the AR spectral density estimate can be written in terms of the parameter estimate vector $\hat{\underline{\theta}}$ as

$$
\hat{\phi}_X(\lambda, \hat{\underline{\theta}}) = \frac{\hat{\sigma}_\varepsilon^2}{2\pi \hat{A}^p(e^{i\lambda}) \hat{A}^p(e^{-i\lambda})} \tag{23}
$$

where the estimate $\hat{A}^p(e^{i\lambda})$ is formed by substituting the AR parameter estimates of (11) into (2) and evaluating at $z = e^{i\lambda}$ and $\hat{\sigma}_\varepsilon^2$ is estimated using (12). We now state and prove the main result of the document.

Theorem 3: Under Assumptions A and B the AR spectral density estimate $\hat{\phi}_X(\lambda, \hat{\theta})$ converges in distribution to a zero mean normal random variable, that is

$$
N^{\frac{1}{2}}[\hat{\phi}_X(\lambda, \hat{\theta}) - \phi_X(\lambda, \theta)] \xrightarrow[N\to\infty]{\mathscr{L}} N(0, \underline{\rho}^T(\lambda) \, \underline{\Sigma} \, \underline{\rho}(\lambda)) \tag{24}
$$

where $\underline{\rho}(\lambda)$ is a gradient vector given by

$$
\underline{\rho}^T(\lambda) = \left[ \frac{\partial \phi_X(\lambda, \underline{\theta})}{\partial \sigma_\varepsilon^2}, \frac{\partial \phi_X(\lambda, \underline{\theta})}{\partial a_1}, \ldots, \frac{\partial \phi_X(\lambda, \underline{\theta})}{\partial a_p} \right] .
$$

$\underline{\text{Proof}}$: By Theorem 2 we have that

$$N^{\frac{1}{2}}(\hat{\underline{\theta}} - \underline{\theta}) \xrightarrow[N\to\infty]{\mathscr{L}} N_{p+1}(\underline{0}, \underline{\Sigma}).$$

Since the function $\phi_X(\lambda, \underline{\theta})$ is totally differentiable with respect to the vector $\underline{\theta}$ the desired result follows directly by a convergence theorem of Rao (reference 6). □

By the result of Theorem 3 we see that from observations of the noise corrupted AR series $Y_n$, through the use of the high-order (Y-W) equations, we can form a weakly consistent spectral estimate for the nonnoise corrupted series $X_n$; the resulting spectral estimate is asymptotically normal with limiting variance $(1/N)\ \underline{\rho}^T(\lambda)\ \underline{\Sigma}\ \underline{\rho}(\lambda)$. We now express $\underline{\rho}^T(\lambda)\ \underline{\Sigma}\ \underline{\rho}(\lambda)$ in terms of previously defined terms.

Let the gradient vector $\underline{\rho}^T(\lambda)$ be written as

$$\underline{\rho}^T(\lambda) \triangleq [b(\lambda), \underline{B}^T(\lambda)]$$

where

$$b(\lambda) \triangleq \frac{\partial\phi_X(\lambda, \underline{\theta})}{\partial\sigma_\varepsilon^2} = \frac{1}{2\pi A^p(e^{i\lambda})A^p(e^{-i\lambda})} \tag{25}$$

and

$$\underline{B}^T(\lambda) \triangleq \left[\frac{\partial\phi_X(\lambda, \underline{\theta})}{\partial a_1}, \frac{\partial\phi_X(\lambda, \underline{\theta})}{\partial a_2}, \cdots, \frac{\partial\phi_X(\lambda, \underline{\theta})}{\partial a_p}\right]$$

$$= \phi_X(\lambda, \underline{\theta}) \left[\mathrm{Re}\left\{\frac{e^{i\lambda}}{A(e^{i\lambda})}\right\}, \mathrm{Re}\left\{\frac{e^{i2\lambda}}{A(e^{i\lambda})}\right\}, \cdots, \mathrm{Re}\left\{\frac{e^{ip\lambda}}{A(e^{i\lambda})}\right\}\right]. \tag{26}$$

We previously defined the matrix $\underline{\Sigma}$ by

$$\underline{\Sigma} \triangleq \left[\begin{array}{c|c} V^2 & \underline{C}^T \\ \hline \underline{C} & \underline{\Phi} \end{array}\right] \tag{27}$$

thus, we can write

$$\underline{\rho}^T(\lambda)\ \underline{\Sigma}\ \underline{\rho}(\lambda) = b^2(\lambda)V^2 + b(\lambda)\ \underline{B}^T(\lambda)\ \underline{C} + b(\lambda)\ \underline{C}^T\ \underline{B}(\lambda) + \underline{B}^T(\lambda)\ \underline{\Phi}\ \underline{B}(\lambda)\ . \tag{28}$$

In the next section, (28) will be evaluated for the first order AR series.

## IV. EXAMPLE

First Order Autoregressive (Markov) Series

The first order AR series ($p = 1$) is given by

$$X_n = a\ X_{n-1} + \varepsilon_n \tag{29}$$

where the parameter a must satisfy the condition $-1<a<1$, i.e., Assumption B, for the series to be stationary. For this first order example the covariance sequence is expressed by

$$r_k = \frac{\sigma_\varepsilon^2\ a^{|k|}}{(1 - a^2)} \qquad k = 0, 1, \cdots \tag{30}$$

and the spectral density is

$$\phi_X(\lambda, \underline{\theta}) = \frac{\sigma_\varepsilon^2}{2\pi[a^2 - 2a \cos \lambda + 1]}$$

where $\underline{\theta}^T = (\sigma_\varepsilon^2, a)$.

Given observations of the noise corrupted version of the AR series, the variance of the spectral density estimate, as given by (28) can be evaluated in terms of the parameter, a, and the variances $\sigma_\varepsilon^2$ and $\sigma_W^2$. For the first order case, $p = 1$, we have

$$\lim_{N\to\infty} N \operatorname{var}\{\hat{\phi}_X(\lambda, \hat{\underline{\theta}})\} = b^2(\lambda)V^2 + 2b(\lambda)B(\lambda)C + B^2(\lambda)\Phi \tag{31}$$

where

$$b(\lambda) = \frac{1}{2\pi\ A(e^{i\lambda})A(e^{-i\lambda})} = \frac{1}{2\pi[a^2 - 2a\cos\lambda + 1]} \tag{32}$$

and

$$B(\lambda) = \frac{2\phi_X(\lambda, \underline{\theta})(\cos\lambda - a)}{A(e^{i\lambda})A(e^{-i\lambda})} = \frac{2\sigma_\varepsilon^2(\cos\lambda - a)}{2\pi[a^2 - 2a\cos\lambda + 1]^2} \quad . \tag{33}$$

Using (20) and (30) we can evaluate $V^2$ for the first order case

$$V^2 = \sigma_w^4\left\{\frac{a^4 - 3a^2 + 1}{a^2}\right\} + 2\sigma_w^2\sigma_\varepsilon^2\left\{\frac{2a^4 + 4a^2 + 1}{a^2(1 + a^2)}\right\} + \sigma_\varepsilon^4\left\{\frac{-a^4 + 4a^2 + 1}{a^2(1 - a^2)}\right\} . \tag{34}$$

Using (21) and (30) we evaluate C and get

$$C = \frac{(1 - a^2)}{a}\left(\sigma_\varepsilon^2 + \frac{\sigma_w{}^4(1 - a^2)}{\sigma_\varepsilon^2}\right) \tag{35}$$

and by using (22) and (30) we get

$$\underline{\Phi} = \frac{(1 - a^2)}{a^2}\left\{1 + (\sigma_w^2/\sigma_\varepsilon^2)(1 - a^2) + (\sigma_w^2/\sigma_\varepsilon^2)^2\ (1 - a^4)\right\} . \tag{36}$$

Substituting (34), (35), and (36) into (31) yields

$$\lim_{N\to\infty} N \operatorname{var}\{\hat{\phi}_X(\lambda, \underline{\hat{\theta}})\} = \frac{\sigma_\varepsilon^2}{(2\pi)^2[a^2 - 2a\cos\lambda + 1]^2}\left[\sigma_\varepsilon^2\left\{\frac{-a^4 + 4a^2 + 1}{a^2(1 - a^2)}\right\}\right.$$

$$\left. + 2\sigma_W^2\left\{\frac{2a^4 + 4a^2 + 1}{a^2(1 + a^2)}\right\} + \frac{\sigma_W^4}{\sigma_\varepsilon^2}\left\{\frac{a^4 - 3a^2 + 1}{a^2}\right\}\right]$$

$$+ \frac{2\sigma_\varepsilon^2(\cos\lambda - a)}{(2\pi)^2[a^2 - 2a\cos\lambda + 1]^3}\ \frac{(1 - a^2)}{a}\ \left[\sigma_\varepsilon^2 + (\sigma_W^4/\sigma_\varepsilon^2)(1 - a^2)\right]$$

$$+ \frac{4\sigma_\varepsilon^4(\cos\lambda - a)^2}{(2\pi)^2[a^2 - 2a\cos\lambda + 1]^4}\ \frac{(1 - a^2)}{a^2}\ \left[1 + \left\{\sigma_W^2/\sigma_\varepsilon^2\right\}(1 - a^2)\right.$$

$$\left. + \left\{\sigma_W^2/\sigma_\varepsilon^2\right\}^2 (1 - a^4)\right] . \tag{37}$$

We see that the limiting variance expression is composed of three major terms: the first term represents the contribution due to variation in estimating $\sigma_\varepsilon^2$, the second term represents the contribution due to the cross-covariance between $\sigma_\varepsilon^2$ and a, and the third term represents the contribution due to variation in estimating the parameter a.

Even for the first-order AR series we see from (37) that the limiting variance expression is a complicated function of the parameters. To provide some insight into the relationship between the variance and the parameters a, $\sigma_\varepsilon^2$, $\sigma_W^2$ and $\lambda$, (37) was evaluated for a few parameter values.

Figure 1 shows the spectral density estimate variance plotted as a function of signal-to-noise ratio (SNR). The AR series used is given by

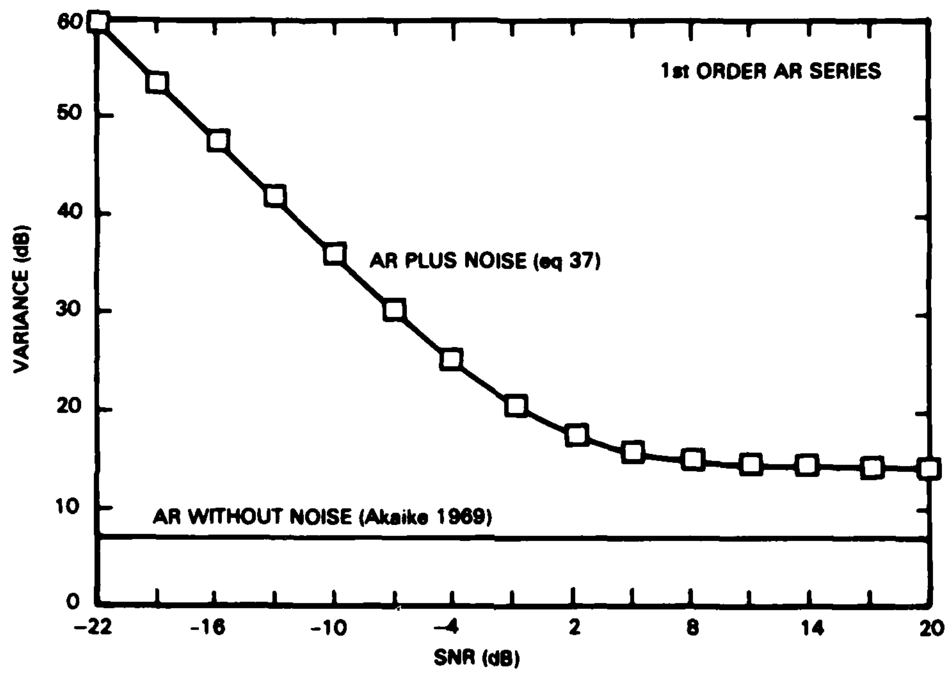


Figure 1. Spectral estimate variance vs signal-to-noise ratio. AR parameter equals 0.5, $\lambda = 0$ radians.

$$X_n = a X_{n-1} + \varepsilon_n \tag{38}$$

with a = 0.5; low pass spectral density. The variance (37) was evaluated for $\lambda = 0$, $\sigma_\varepsilon^2 = 1$ and $\sigma_w^2$ set to achieve the indicated SNRs. We see that the variance decreases monotonically with increasing SNR to a value of 14 dB at a SNR of 8 dB. Also plotted is the variance obtained by Akaike (reference 7) for the first-order AR series without noise as indicated by the horizontal line at about 7 dB. Note that the AR plus noise case variance, at high SNR, does not asymptotically approach the no noise variance. This is the case because the high-order Y-W equation estimates of the AR parameter used for the AR plus noise case produces a larger parameter estimate variance than that of the conventional Y-W equation estimate. That is, for the first-order AR series with a = 0.5 we have from Akaike (reference 7) that

$$\lim_{N\to\infty} N \, \mathrm{var}(\hat{a}) = (1 - a^2) = 0.75$$

and for the AR plus noise case we have from the third term on the right side of (37) with $\sigma_w^2 = 0$ that

$$\lim_{N\to\infty} N \, \mathrm{var}(\hat{a}) = (1 - a^2)/a^2 = 3.0 \quad .$$

In figure 2 the estimated variance is plotted as a function of SNR for $\lambda = 0$ for two values of the AR parameter, a = 0.1 and a =0.8. We see the same monotonic decrease with increasing SNR for both cases as in figure 1. The asymptotic limit for the a = 0.8 case is about 15 dB greater than that for a = 0.1. Thus indicating that as the spectral density bandwidth decreases the spectral estimate variance increases. In figure 3 we have spectral estimate variance plotted as a function of frequency for three values of the AR parameter a = 0.1, a = 0.5, and a = 0.8. We see that for the two narrower bandwidth cases, a = 0.5 and 0.8, the variance decreases monotonically with increasing frequency but for the wider bandwidth case a = 0.1 there is little variation with frequency over the range evaluated.

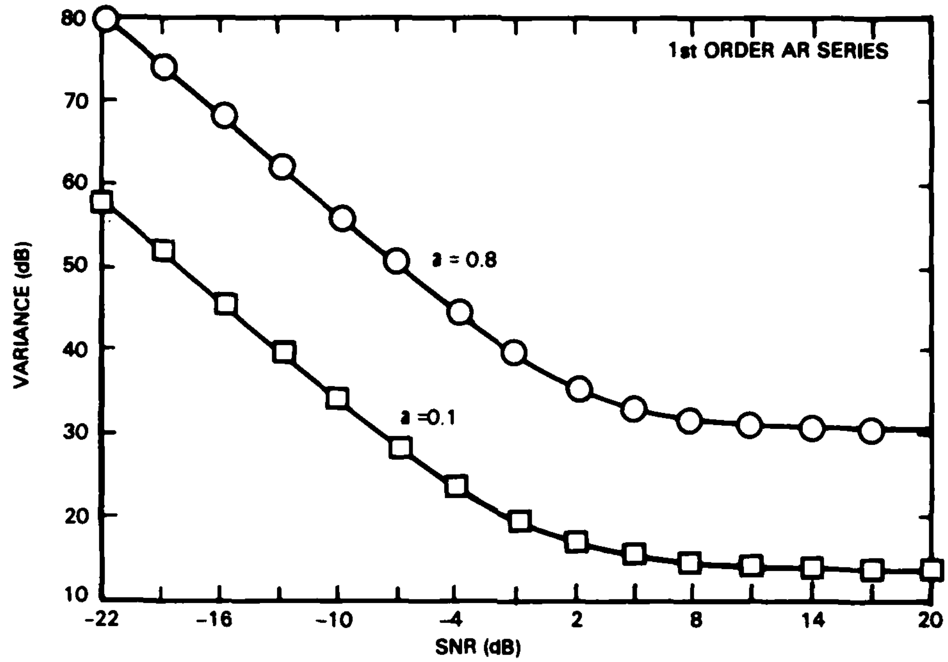


Figure 2. Spectral estimate variance vs signal-to-noise ratio.
AR parameter equals 0.1 and 0.8, λ = 0 radians.

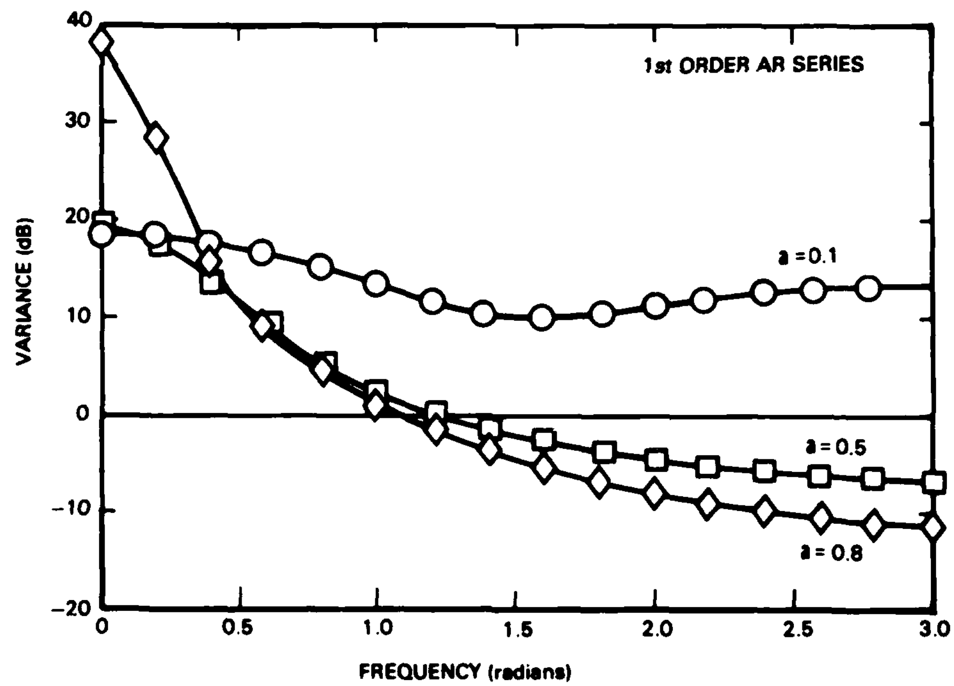


Figure 3. Spectral estimate variance vs frequency (radians).
Signal-to-noise ratio equals 0 dB; AR parameter equals 0.1, 0.5, and 0.8.

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# VI. APPENDIX

## A. Evaluation of $\underline{\Phi}$:

From Section III we have

$$\underline{\Gamma}_p \, \underline{\Phi} \, \underline{\Gamma}_p^T = 2\pi \int_{-\pi}^{\pi} \underline{D} \, \underline{U}_{11} \, \underline{D}^T \, \phi_Y^2(\lambda) d\lambda. \tag{A-1}$$

We first evaluate the n, $m^{th}$ element of $\underline{D} \, \underline{U}_{11} \, \underline{D}^T$ and have

$$\{\underline{D} \, \underline{U}_{11} \, \underline{D}^T\}_{nm} = \sum_{k=1}^{2p} \sum_{j=1}^{2p} d_{nk} \, u_{kj} \, d_{mj} \qquad n = 1, \cdots, p; \quad m = 1, \cdots, p.$$

By the definition of the matrix $U_{11}$ we have

$$\{\underline{D} \, \underline{U}_{11} \, \underline{D}^T\}_{nm} = \sum_{k=1}^{2p} \sum_{j=1}^{2p} d_{nk} \, e^{ik\lambda} \, d_{mj} \, \{e^{ij\lambda} + e^{-ij\lambda}\}$$

and by the definition of the matrix $\underline{D}$ we get

$$\{\underline{D} \, \underline{U}_{11} \, \underline{D}^T\}_{nm} = A^p(e^{-i\lambda}) \, e^{i(p+n)\lambda} \left\{ A^p(e^{-i\lambda}) \, e^{i(p+m)\lambda} + A^p(e^{i\lambda}) e^{-i(p+m)\lambda} \right\}. \tag{A-2}$$

Substituting (A-2) into (A-1) we get

$$\underline{\Gamma}_p \, \underline{\Phi} \, \underline{\Gamma}_p^T = \underline{T}_1 + \underline{T}_2$$

where

$$\{\underline{T}_1\}_{nm} \triangleq 2\pi \int_{-\pi}^{\pi} A(e^{-i\lambda}) \, A(e^{-i\lambda}) \, e^{i(2p+n+m)\lambda} \, \phi_Y^2(\lambda) d\lambda \tag{A-3}$$

and

$$\{\underline{T}_2\}_{nm} \triangleq 2\pi \int_{-\pi}^{\pi} A(e^{-i\lambda})\ A(e^{i\lambda})\ e^{i(n-m)\lambda}\ \phi_Y^2(\lambda)d\lambda \quad . \tag{A-4}$$

For the AR(p) plus noise process we have that

$$\phi_Y^2(\lambda) = \left\{\sigma_w^4 + \frac{2\sigma_w^2\sigma_\varepsilon^2}{A^p(e^{i\lambda})A^p(e^{-i\lambda})} + \frac{\sigma_\varepsilon^4}{[A^p(e^{i\lambda})A^p(e^{-i\lambda})]^2}\right\} . \tag{A-5}$$

Substituting the expression (A-5) into (A-3) and (A-4) and carrying out the integration we get

$$\{\underline{T}_1\}_{nm} = 0 \qquad\qquad n = 1, \cdots, p; \quad m = 1, \cdots, p$$

and

$$\{\underline{T}_2\}_{nm} = \sigma_w^4 \sum_{j=0}^{p-|n-m|} a_j a_{j+|n-m|} + \sigma_w^2\sigma_\varepsilon^2\ \delta(n-m) + \sigma_\varepsilon^2\ r_{n-m} \quad .$$

Using these results we get

$$\underline{\Phi} = \sigma_\varepsilon^2\ \underline{\Gamma}_p^{-1}\ \underline{\Gamma}_0 \left(\underline{\Gamma}_p^{-1}\right)^T + \sigma_w^2\ \underline{\Gamma}_p^{-1}[\sigma_\varepsilon^2\ \underline{I} + \sigma_w^2\ \underline{Q}]\left(\underline{\Gamma}_p^{-1}\right)^T \tag{A-6}$$

where the matrix $\underline{Q}$ was defined in Section III.

B. Evaluation of $\underline{C}^T$:

From Section III we have

$$\underline{C}^T = 2\pi \int_{-\pi}^{\pi} \underline{H}\ \underline{U}_{01}\ \underline{D}^T\left(\underline{\Gamma}_p^{-1}\right)^T\ \phi_Y^2(\lambda)d\lambda \tag{A-7}$$

with $\underline{H}$ defined in Lemma 2. Substituting the expression for $\underline{H}$ into (A-7) we have

$$\underline{C}^T = -2\pi \int_{-\pi}^{\pi} [-1,\ \underline{a}^T,\ \underline{0}]\ \underline{U}_{01}\ \underline{D}^T \left(\underline{\Gamma}_p^{-1}\right)^T \phi_Y^2(\lambda)d\lambda$$

$$- 2\pi(1/a_p) \int_{-\pi}^{\pi} [\underline{\tilde{a}}^T,\ -1,\ \underline{0}]\ \underline{U}_{01}\ \underline{D}^T \left(\underline{\Gamma}_p^{-1}\right)^T \phi_Y^2(\lambda)d\lambda \qquad \text{(A-8)}$$

$$\triangleq \underline{T}_1 + \underline{T}_2 \ .$$

We first examine the contribution due to $\underline{T}_1$. By the definition of the matrices $\underline{U}_{01}$ and $\underline{D}$ we can write the $l^{th}$ element of $[-1,\ \underline{a}^T,\ \underline{0}]\ \underline{U}_{01}\ \underline{D}^T$ by

$$\{[-1,\ \underline{a}^T,\ \underline{0}]\ \underline{U}_{01}\ \underline{D}^T\}_l = \sum_{k=0}^{p} a_k\ e^{ik\lambda} \sum_{j=1}^{p+1} a_{p-j+l}\ \{e^{ij\lambda} + e^{-ij\lambda}\} \qquad l = 1,2,\cdots,p$$

$$= A^p(e^{i\lambda})A^p(e^{-i\lambda})\ e^{i(p+l)\lambda} + A^p(e^{i\lambda})A^p(e^{i\lambda})e^{-i(p+l)\lambda} \ . \qquad \text{(A-9)}$$

Thus, by (A-9) and (A-8) we have

$$\{\underline{T}_1\ \underline{\Gamma}_p^T\}_l = -2\pi \int_{-\pi}^{\pi} \{A^p(e^{i\lambda})A^p(e^{-i\lambda})e^{i(p+l)\lambda} + A^p(e^{i\lambda})A^p(e^{i\lambda})e^{-i(p+l)\lambda}\}\phi_Y^2(\lambda)d\lambda$$

$$\triangleq \{\underline{S}_1\}_l + \{\underline{S}_2\}_l \ . \qquad \text{(A-10)}$$

Evaluating $\underline{S}_1$ we get

$$\{\underline{S}_1\}_l = -2\pi \int_{-\pi}^{\pi} A^p(e^{i\lambda})A^p(e^{-i\lambda})e^{i(p+l)\lambda}\ \phi_Y^2(\lambda)d\lambda$$

Using (A-5) and carrying out the integration yields

$$\{\underline{S}_2\}_l = -\sigma_\varepsilon\ r_{p+l} \ . \qquad \text{(A-11)}$$

For $\{\underline{S}_2\}_l$

$$\{\underline{S}_2\}_l = -2\pi \int_{-\pi}^{\pi} A^p(e^{i\lambda})A^p(e^{i\lambda})e^{-i(p+l)\lambda}\ \phi_Y^2(\lambda)d\lambda$$

and by (A-5) we get

$$\{\underline{S}_2\}_l = -\sigma_w^4 \sum_{j=l}^{p} a_j a_{p+l-j} \quad . \tag{A-12}$$

For $\underline{T}_2$ we have by the definition of $\underline{U}_{01}$ and $\underline{D}$ that the $l^{th}$ element of $(1/a_p)\ [\underline{\tilde{a}}^T, -1, \underline{0}]\ \underline{U}_{01}\ \underline{D}^T$ can be expressed as

$$(1/a_p)\ \{[\underline{\tilde{a}}^T, -1, \underline{0}]\ \underline{U}_{01}\ \underline{D}^T\}_l = \sum_{k=0}^{p} a_{p-k} \sum_{j=l}^{p+l} a_{p-j+l}\ \{e^{i(k+j)\lambda} + e^{i(k-j)\lambda}\}$$

$$= A^p(e^{-i\lambda})A^p(e^{-i\lambda})e^{i(2p+l)\lambda} + A^p(e^{-i\lambda})A^p(e^{i\lambda})e^{-il\lambda}$$

and

$$a_p \left\{\underline{T}_2\ \underline{\Gamma}_p^T\right\}_l = -2\pi \int_{-\pi}^{\pi} \{A^p(e^{-i\lambda})A^p(e^{-i\lambda})e^{i(2p+l)\lambda} + A^p(e^{-i\lambda})A^p(e^{i\lambda})e^{-il\lambda}\}\phi_Y^2(\lambda)d\lambda$$

$$\triangleq \{\underline{S}_3\}_l + \{\underline{S}_4\}_l \quad . \tag{A-13}$$

Evaluating $\underline{S}_3$ using (A-5) for $\phi_Y^2(\lambda)$ we get

$$\{\underline{S}_3\}_l = 0 \qquad l = 1,\ 2,\ \cdots,\ p \tag{A-14}$$

and evaluating $\underline{S}_4$ we get

$$\{S_4\}_l = -\sigma_w^4 \sum_{j=0}^{p-l} a_j a_{j+l} - \sigma_\varepsilon^2 r_l \quad . \tag{A-15}$$

Define the $l^{th}$ element of the vector $\underline{P}^T$ by

$$\{\underline{P}^T\}_l = \left\{\underline{I}_1\underline{\Gamma}_p^T\right\}_l + \left\{\underline{I}_2\ \underline{\Gamma}_p^T\right\}_l$$

then using (A-11) and (A-12) in (A-10) and (A-14) and (A-15) in (A-13) we get

$$\{\underline{P}^T\}_l = -\sigma_\varepsilon^2 r_{p+l} - \sigma_w^4 \sum_{j=1}^{p} a_j a_{p+l-j} - (1/a_p)\sigma_w^4 \sum_{j=0}^{p-l} a_j a_{j+l} - (1/a_p)\sigma_\varepsilon^2 r_l$$

$$l = 1, 2, \cdots, p.$$

It follows that

$$\underline{C}^T = \underline{P}^T\left(\underline{\Gamma}_p^{-1}\right)^T \quad .$$

C. Evaluation of $V^2$:

From Section III we have

$$V^2 = 2\pi \int_{-\pi}^{\pi} \underline{H}\ \underline{U}_{00}\ \underline{H}^T\ \phi_Y^2\ (\lambda)d\lambda \tag{A-16}$$

where $\underline{H}$ was defined in Lemma 2. Using the expression for $\underline{H}$ in (A-16) we get

$$\begin{aligned} V^2 = {} & 2\pi \int_{-\pi}^{\pi} [-1,\ \underline{a}^T,\ \underline{0}]\ U_{00}[-1,\ \underline{a}^T,\ \underline{0}]^T\ \phi_Y^2(\lambda)d\lambda \\ & + 2\pi \int_{-\pi}^{\pi} [-1,\ \underline{a}^T,\ \underline{0}]\ U_{00}[\underline{\tilde{a}}^T,\ -1,\ \underline{0}](1/a_p)\phi_Y^2(\lambda)d\lambda \\ & + 2\pi \int_{-\pi}^{\pi} (1/a_p)\ [\underline{\tilde{a}}^T,\ -1,\ \underline{0}]\ U_{00}[-1,\ \underline{a}^T,\ \underline{0}]^T\ \phi_Y^2(\lambda)d\lambda \\ & + 2\pi \int_{-\pi}^{\pi} (1/a_p)^2\ [\underline{\tilde{a}}^T,\ -1,\ \underline{0}]\ U_{00}[\underline{\tilde{a}}^T,\ -1,\ \underline{0}]\phi_Y^2(\lambda)d\lambda \\ \triangleq {} & T_1 + T_2 + T_3 + T_4 \quad . \end{aligned}$$

By the definition of the matrix $U_{oo}$ we have for $T_1$

$$T_1 = 2\pi \int_{-\pi}^{\pi} \sum_{k=0}^{p} \sum_{j=0}^{p} a_k a_j e^{ik\lambda}(e^{ij\lambda} + e^{-ij\lambda})\phi_Y^2(\lambda)d\lambda$$

$$= 2\pi \int_{-\pi}^{\pi} [A^p(e^{i\lambda})A^p(e^{i\lambda}) + A^p(e^{i\lambda})A^p(e^{-i\lambda})]\phi_Y^2(\lambda)d\lambda \quad .$$

Using (A-5) for $\phi_Y^2(\lambda)$ and performing the integrations we get

$$T_1 = \sigma_w^4 + \sigma_\varepsilon^4 + \sigma_w^2\sigma_\varepsilon^2 + \sigma_\varepsilon^2 r_0 + \sigma_w^4 \sum_{j=0}^{p} a_j^2 + 2\sigma_w^2\sigma_\varepsilon^2 \Big/ \sum_{j=0}^{p} a_j^2 \quad .$$

For $T_2$ we have

$$T_2 = 2\pi \int_{-\pi}^{\pi} (1/a_p) \sum_{j=0}^{p} \sum_{K=0}^{p} a_j a_{p-k}[e^{i(k+j)\lambda} + e^{i(k-j)\lambda}]\phi_Y^2(\lambda)d\lambda$$

again using (A-5) and performing the integrations we get

$$T_2 = -\sigma_w^4 + (1/a_p)\sigma_\varepsilon^2 r_p + (1/a_p)\sigma_w^4 \sum_{j=0}^{p} a_j a_{p-j} \quad .$$

For $T_3$ we have

$$T_3 = 2\pi \int_{-\pi}^{\pi} (1/a_p) \sum_{j=0}^{p} \sum_{k=0}^{p} a_{p-j} a_k \, e^{ik\lambda}(e^{ij\lambda} + e^{-ij\lambda})$$

$$= 2\pi \int_{-\pi}^{\pi} (1/a_p)[A^p(e^{i\lambda})A^p(e^{-i\lambda})e^{ip\lambda} + A(e^{i\lambda})A(e^{i\lambda})e^{-ip\lambda}]\phi_Y^2(\lambda)d\lambda \quad .$$

Using (A-5) and performing the integrations we get

$$T_3 = -\sigma_w^4 + (1/a_p)\sigma_\varepsilon^2 r_p + (1/a_p)\sigma_w^4 \sum_{j=0}^{p} a_j a_{p-j} + 2\sigma_w^2 \quad .$$

For $T_4$ we have

$$T_4 = 2\pi \int_{-\pi}^{\pi} (1/a_p)^2 \sum_{j=0}^{p}\sum_{k=0}^{p} a_{p-j}a_{p-k}\, e^{ik\lambda}(e^{ij\lambda} + e^{-ij\lambda})\phi_Y^2(\lambda)d\lambda$$

$$= 2\pi \int_{-\pi}^{\pi} (1/a_p)^2[A^p(e^{-i\lambda})A^p(e^{-i\lambda})e^{i2p\lambda} + A^p(e^{-i\lambda})A^p(e^{i\lambda})]\phi_Y^2(\lambda)d\lambda \quad .$$

Using (A-5) and performing the integrations we get

$$T_4 = (1/a_p)^2 \left\{\sigma_w^4 \sum_{j=0}^{p} a_j^2 + 2\sigma_w^2\,\sigma_\varepsilon^2 + \sigma_\varepsilon^2(r_0 - \sigma_w^2)\right\} \quad .$$

Thus,

$$V^2 = \sigma_w^4\left\{\sum_{j=0}^{p} a_j^2 + (2/a_p)\sum_{j=0}^{p} a_j a_{p-j} - 1 + (1/a_p)^2 \sum_{j=0}^{p} a_j^2\right\}$$

$$+ \sigma_w^2\sigma_\varepsilon^2\left\{\left(2\Big/\sum_{j=0}^{p} a_j^2\right) + 3 + (1/a_p)^2\right\}$$

$$+ \sigma_\varepsilon^2\{\sigma_\varepsilon^2 + r_0 + (2/a_p)r_p + (1/a_p)^2\, r_0\}$$

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